

REALISING HIGHER CLUSTER CATEGORIES OF DYNKIN TYPE AS STABLE MODULE CATEGORIES

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ABSTRACT. We show that the stable module categories of certain selfinjective algebras of finite representation type having tree class A_n , D_n , E_6 , E_7 or E_8 are triangulated equivalent to u -cluster categories of the corresponding Dynkin type. The proof relies on the “Morita” theorem for u -cluster categories by Keller and Reiten, along with the recent computation of Calabi-Yau dimensions of stable module categories by Dugas.

1. INTRODUCTION

This paper deals with two types of categories: *Stable module categories of selfinjective algebras* and *u -cluster categories*. They both originate in representation theory, and we will establish a connection between the two by showing that some stable module categories are, in fact, u -cluster categories.

Stable module categories are classical objects of representation theory. They arise from categories of finitely generated modules through the operation of dividing by the ideal of homomorphisms which factor through a projective module. The stable module category of a finite dimensional selfinjective algebra has the appealing property that it is triangulated; this has been very useful not least in group representation theory.

Cluster categories and the more general *u -cluster categories* which are parametrised by the natural number u were introduced over the last few years in a number of beautiful papers: [5], [7], [19], [25], and [26]. The idea is to provide categorifications of the theory of cluster algebras and higher cluster complexes as introduced in [10] and [11]. If Q is a finite quiver without loops and oriented cycles, then the u -cluster category of type Q over a field k is defined by considering the bounded derived category of the path algebra kQ and taking the orbit category of a certain autoequivalence; see Section 2 for details. A u -cluster category is triangulated; this non-trivial fact was established in [19].

The introduction of cluster categories and u -cluster categories has created a rush of activity which has turned these categories into a major item of contemporary representation theory. This is due not least to the advent of cluster tilting theory in [6], which provides a long awaited generalization of classical tilting theory making it possible to tilt at any vertex of the quiver of a hereditary algebra, not just at sinks and sources.

In this paper, we will show that a number of stable module categories of selfinjective algebras are, in fact, u -cluster categories.

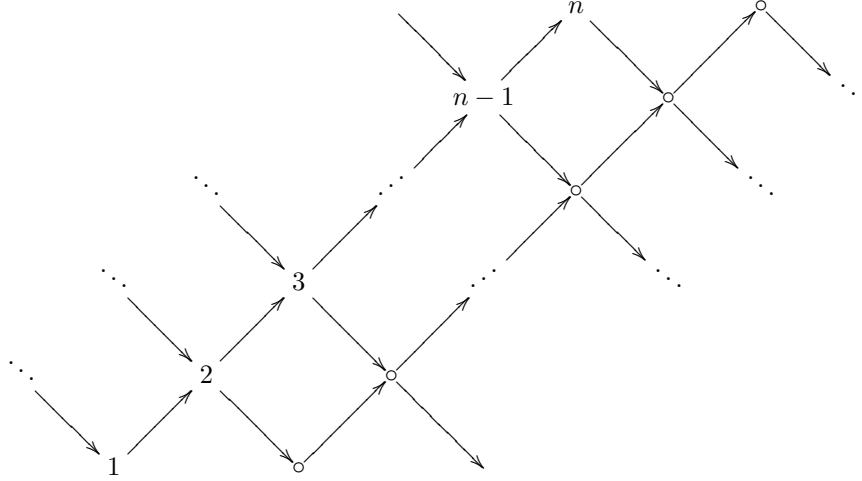
To be precise, we will look at stable module categories of selfinjective algebras of finite representation type. By the Riedtmann structure theorem [23] the Auslander-Reiten (AR) quiver of such

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$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n$$

FIGURE 1. The Dynkin quiver A_n FIGURE 2. The repetitive quiver $\mathbb{Z}A_n$

a category has tree class of Dynkin type A_n , D_n , E_6 , E_7 , or E_8 . We illustrate in type A what this means. Consider the Dynkin quiver in Figure 1 which, by abuse of notation, we will often denote by A_n , and its repetitive quiver $\mathbb{Z}A_n$ shown in Figure 2. For a selfinjective algebra to have finite representation type and tree class A_n means that the AR quiver of its stable module category is a non-trivial quotient of $\mathbb{Z}A_n$ by an admissible group of automorphisms. In type A , in such a quotient, two vertical lines on the quiver are identified, and this gives either a cylinder or a Möbius band. According to this dichotomy, the algebra belongs to one of two well understood classes: the Nakayama algebras and the Möbius algebras.

For tree classes D_n and E_6 , E_7 , E_8 , the shapes of the stable AR quivers are obtained in a very similar fashion; more details on the precise shapes are given in Section 5 for type D and Section 6 for type E .

Now, u -cluster categories of Dynkin types A_n , D_n , E_6 , E_7 , and E_8 also have AR quivers which are either cylinders or Möbius bands; see Section 2 for details. One of the aims of this paper is to show that this resemblance is no coincidence.

For stating the main results of the paper we have to deal with the various Dynkin types separately.

Let us start with Dynkin type A . For integers $N, n \geq 1$, let $B_{N,n+1}$ denote the Nakayama algebra defined as the path algebra of the circular quiver with N vertices and all arrows pointing in the same direction modulo the ideal generated by all paths of length $n+1$. Moreover, for integers $p, s \geq 1$, let $M_{p,s}$ denote the corresponding Möbius algebra (for the definition of these algebras by quivers and relations, see Section 4.b).

The following is our first main result which gives a complete list of those u -cluster categories of type A which are triangulated equivalent to stable module categories of selfinjective algebras.

Theorem A (Realising u -cluster categories of type A).

- (i) Let $u \geq 2$ be an even integer and let $n \geq 1$ be an integer. Set $N = \frac{u}{2}(n+1) + 1$. Then the u -cluster category of type A_n is equivalent as a triangulated category to the stable module category $\text{stab } B_{N,n+1}$.
- (ii) Let $u \geq 1$ be an odd integer and let $p, s \geq 1$ be integers for which $s(2p+1) = u(p+1) + 1$. Then the u -cluster category of type A_{2p+1} is equivalent as a triangulated category to the stable module category $\text{stab } M_{p,s}$.

We next consider Dynkin types D and E . The theory becomes more intricate than in type A . While two types of selfinjective algebras occurred in type A , we will show that three types of algebras occur in type D , and two in type E . More precisely, in Asashiba's notation from [1, appendix], they are the algebras $(D_n, s, 1)$, $(D_n, s, 2)$, and $(D_{3m}, \frac{s}{3}, 1)$ in type D , and $(E_n, s, 1)$, $n = 6, 7, 8$, and $(E_6, s, 2)$ in type E .

Specifically, we show the following main results.

Theorem D (Realising u -cluster categories of type D). Let m, n, u be integers with $u \geq 1$.

- (i) Suppose that $n \geq 4$ and $u \equiv -2 \pmod{2n-3}$.

Then the u -cluster category of type D_n is equivalent as a triangulated category to the stable module category

$$\begin{cases} \text{stab}(D_n, \frac{u(n-1)+1}{2n-3}, 1) & \text{if } n \text{ or } u \text{ is even,} \\ \text{stab}(D_n, \frac{u(n-1)+1}{2n-3}, 2) & \text{if } n \text{ and } u \text{ are odd.} \end{cases}$$

- (ii) Suppose that $m \geq 2$ and $u \equiv -2 \pmod{2m-1}$ but $u \not\equiv -2 \pmod{6m-3}$. Moreover suppose that not both m and u are odd. Then the u -cluster category of type D_{3m} is equivalent as a triangulated category to the stable module category $\text{stab}(D_{3m}, \frac{s}{3}, 1)$ where $s = \frac{u(3m-1)+1}{2m-1}$.

Theorem E (Realising u -cluster categories of type E). Let $u \geq 1$ be an integer.

- (i) If $u \equiv -2 \pmod{11}$ then the u -cluster category of type E_6 is equivalent as a triangulated category to the stable module category

$$\begin{cases} \text{stab}(E_6, \frac{6u+1}{11}, 1) & \text{if } u \text{ is even,} \\ \text{stab}(E_6, \frac{6u+1}{11}, 2) & \text{if } u \text{ is odd.} \end{cases}$$

- (ii) If $u \equiv -2 \pmod{17}$ then the u -cluster category of type E_7 is equivalent as a triangulated category to the stable module category $\text{stab}(E_7, \frac{9u+1}{17}, 1)$.
- (iii) If $u \equiv -2 \pmod{29}$ then the u -cluster category of type E_8 is equivalent as a triangulated category to the stable module category $\text{stab}(E_8, \frac{15u+1}{29}, 1)$.

The proofs of the above theorems rely on the seminal ‘‘Morita theorem’’ for u -cluster categories established by Keller and Reiten in [20]. The idea is to show that the stable module categories of the relevant selfinjective algebras have very strong formal properties in terms of their Calabi-Yau dimensions and u -cluster tilting objects. More precisely, the Keller-Reiten structure theorem states the following. Consider a Hom finite triangulated category of algebraic origin (e.g. the stable module category of a selfinjective algebra). Assume that it has Calabi-Yau dimension $u+1$ and possesses a u -cluster tilting object T which has hereditary endomorphism algebra H and also satisfies $\text{Hom}(T, \Sigma^{-i}T) = 0$ for $i = 1, \dots, u-1$ where Σ is the suspension functor. Then this category is triangulated equivalent to the u -cluster category of H .

Theorems A, D and E were already stated in our earlier preprints [15], [16] which were later withdrawn. Unfortunately there was a mistake in [15], pointed out to us by Alex Dugas, in connection with the Calabi-Yau dimensions, and this meant there was a gap in the proofs of the main results of [15] and [16].

In this paper we circumvent the problem and thereby provide correct proofs of the above theorems. This is achieved by using a recent paper of Dugas [8] in which he computes the Calabi-Yau dimensions for stable module categories of selfinjective algebras of finite representation type.

The paper is organized as follows: Section 2 collects the properties of u -cluster categories of Dynkin types ADE which we will need. Section 3 is a remark on u -cluster tilting objects in stable module categories. Section 4 considers Dynkin type A and proves Theorem A. This is split into subsections 4.a and 4.b dealing with Nakayama algebras and Möbius algebras; these two situations correspond to parts (i) and (ii) of Theorem A. Sections 5 and 6 similarly consider Dynkin types D and E and prove Theorems D and E.

Throughout, k is an algebraically closed field, A is a selfinjective k -algebra, $\text{mod } A$ denotes the category of finitely generated right- A -modules, and $\text{stab } A$ denotes the stable category of finitely generated right- A -modules.

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2. u -CLUSTER CATEGORIES

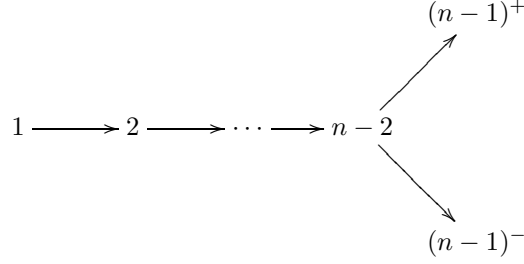
This section collects the properties of u -cluster categories which we will need.

Let Q be a finite quiver without loops and oriented cycles. Consider the path algebra kQ and let $D^f(kQ)$ be the derived category of bounded complexes of finitely generated right- kQ -modules. See [14] for background on $D^f(kQ)$ and [22] for additional information on AR theory and Serre functors.

If $u \geq 1$ is an integer, then the u -cluster category of type Q is defined as $D^f(kQ)$ modulo the functor $\tau^{-1}\Sigma^u$, where τ is the AR translation of $D^f(kQ)$ and Σ the suspension. In other words, the u -cluster category is the orbit category for the action of $\tau^{-1}\Sigma^u$ on the category $D^f(kQ)$. Denote the u -cluster category of type Q by C .

It follows from [19, sec. 4, thm.] that C admits a structure of triangulated category in a way such that the canonical functor $D^f(kQ) \rightarrow C$ is triangulated.

The category C has Calabi-Yau dimension $u + 1$ by [20, sec. 4.1]. That is, $n = u + 1$ is the smallest non-negative integer such that Σ^n , the n th power of the suspension functor, is the Serre functor of C .

FIGURE 3. The Dynkin quiver D_n

The category \mathcal{C} has the same objects as the derived category $D^f(kQ)$, so in particular, kQ is an object of \mathcal{C} . In fact, by [20, sec. 4.1] again, kQ is a u -cluster tilting object of \mathcal{C} , cf. [18, sec. 3]. That is,

- (i) $\text{Hom}_{\mathcal{C}}(kQ, \Sigma t) = \cdots = \text{Hom}_{\mathcal{C}}(kQ, \Sigma^u t) = 0 \Leftrightarrow t \in \text{add } kQ$,
- (ii) $\text{Hom}_{\mathcal{C}}(t, \Sigma kQ) = \cdots = \text{Hom}_{\mathcal{C}}(t, \Sigma^u kQ) = 0 \Leftrightarrow t \in \text{add } kQ$.

Recall that $\text{add } kQ$ denotes the full subcategory of \mathcal{C} consisting of direct summands of (finite) direct sums of copies of kQ .

The endomorphism ring $\text{End}_{\mathcal{C}}(kQ)$ is kQ itself.

2.a. u -cluster categories of Dynkin type A . Let Q be a Dynkin quiver of type A_n for an integer $n \geq 1$. This means that the graph obtained from Q by forgetting the orientations of the arrows is a Dynkin diagram of type A_n . Recall that the orientation of Q is not important since for any two orientations the derived categories $D^f(kQ)$ are triangulated equivalent. In the sequel we shall always use the linear orientation as in Figure 1 in the introduction.

By [14, cor. 4.5(i)], the AR quiver of $D^f(kQ)$ is the repetitive quiver $\mathbb{Z}A_n$; see Figure 2 in the introduction. Accordingly, the AR quiver of the u -cluster category \mathcal{C} is $\mathbb{Z}A_n$ modulo the action of $\tau^{-1}\Sigma^u$ by [5, prop. 1.3].

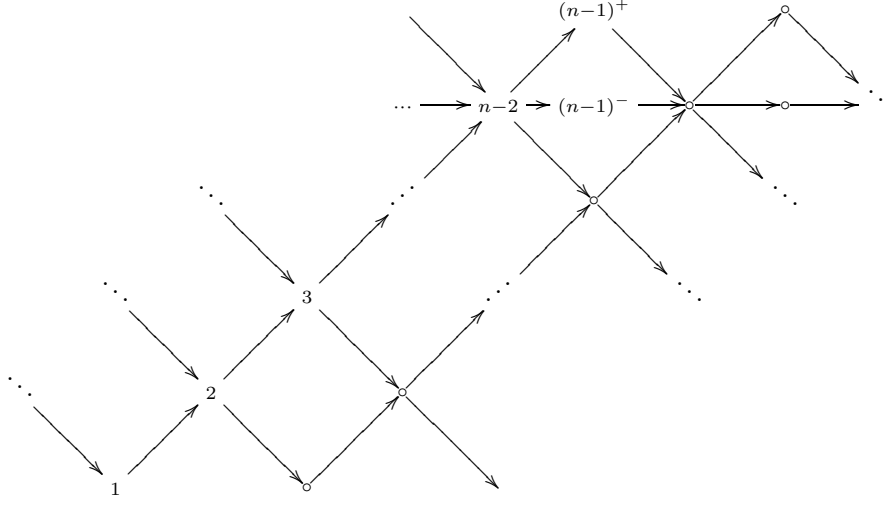
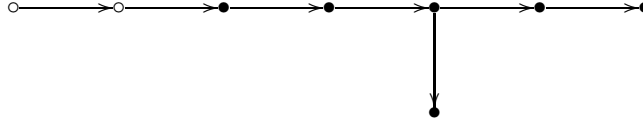
The AR translation τ of $D^f(kQ)$ acts on the quiver by shifting one unit to the left. Both here and below, a unit equals the distance between two vertices which are horizontal neighbours. Hence τ^{-1} acts by shifting one unit to the right.

The suspension Σ of $D^f(kQ)$ acts by reflecting in the horizontal centre line and shifting $\frac{n+1}{2}$ units to the right; see [21, table p. 359]. Note that this shift makes sense for all values of n : If n is even, then the reflection in the horizontal centre line sends a vertex of the quiver to a point midwise between two vertices, and the half integer shift by $\frac{n+1}{2}$ sends this point to a vertex.

It follows that if u is even, then $\tau^{-1}\Sigma^u$ acts by shifting $\frac{u}{2}(n+1) + 1$ units to the right, and if u is odd, then $\tau^{-1}\Sigma^u$ acts by shifting $\frac{u}{2}(n+1) + 1$ units to the right and reflecting in the horizontal centre line.

So if u is even, then the AR quiver of \mathcal{C} has the shape of a cylinder, and if u is odd, then the AR quiver of \mathcal{C} has the shape of a Möbius band.

2.b. u -cluster categories of Dynkin type D . Let Q be a Dynkin quiver of type D_n for an integer $n \geq 4$. Since the orientation of the quiver does not affect the derived category, we can assume that Q has the form in Figure 3. By [14, cor. 4.5(i)], the AR quiver of $D^f(kQ)$ is the

FIGURE 4. The repetitive quiver $\mathbb{Z}D_n$ FIGURE 5. The Dynkin quivers E_6, E_7, E_8

repetitive quiver $\mathbb{Z}D_n$ shown in Figure 4. The AR quiver of the u -cluster category \mathcal{C} is $\mathbb{Z}D_n$ modulo the action of $\tau^{-1}\Sigma^u$ by [5, prop. 1.3].

Again τ^{-1} acts by shifting one unit to the right.

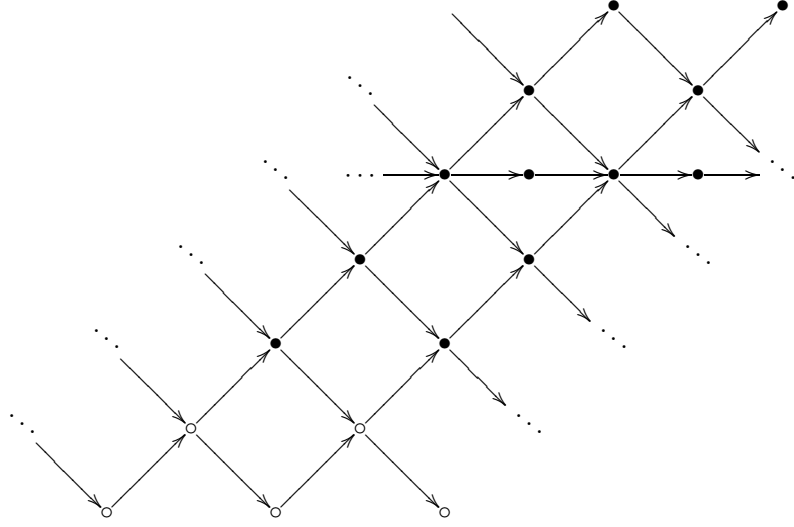
If n is even, then the suspension Σ acts by shifting $n - 1$ units to the right, and if n is odd, then Σ acts by shifting $n - 1$ units to the right and switching each pair of ‘exceptional’ vertices such as $(n - 1)^+$ and $(n - 1)^-$; cf. [21, table p. 359].

It follows that if n or u is even, then $\tau^{-1}\Sigma^u$ acts by shifting $u(n - 1) + 1$ units to the right, and if n and u are both odd, then $\tau^{-1}\Sigma^u$ acts by shifting $u(n - 1) + 1$ units to the right and switching each pair of exceptional vertices.

Accordingly, the AR quiver of the u -cluster category \mathcal{C} has the shape of a cylinder of circumference $u(n - 1) + 1$.

2.c. u -cluster categories of Dynkin type E . Let Q be a Dynkin quiver of type E_n for $n = 6, 7, 8$. We can suppose that Q has the orientation in Figure 5, with the convention that for $n = 6$ the two non-filled vertices and for $n = 7$ the leftmost non-filled vertex (and all arrows incident to them) do not exist. By [14, cor. 4.5(i)], the AR quiver of $D^f(kQ)$ is the repetitive quiver $\mathbb{Z}E_n$ shown in Figure 6. Again, for $n = 6$ and $n = 7$ the bottom two rows and bottom row, respectively, of non-filled vertices do not occur. Note that for $n = 6$ the AR quiver has a symmetry at the central line which does not exist for $n = 7, 8$.

The AR quiver of the u -cluster category \mathcal{C} is $\mathbb{Z}E_n$ modulo the action of $\tau^{-1}\Sigma^u$ by [5, prop. 1.3].

FIGURE 6. The repetitive quivers $\mathbb{Z}E_6, \mathbb{Z}E_7, \mathbb{Z}E_8$

Again τ^{-1} acts by shifting one unit to the right.

If $n = 6$ then the suspension Σ acts by shifting 6 units to the right and reflecting in the central line of the AR quiver. If $n = 7, 8$ then Σ acts by shifting 9, respectively 15 units to the right. See [21, table 1, p. 359].

It follows that the action of $\tau^{-1}\Sigma^u$ is given as follows: for $n = 6$ and u even, by shifting $6u + 1$ units to the right; for $n = 6$ and u odd, by shifting $6u + 1$ units to the right and reflecting in the central line; for $n = 7$, by shifting $9u + 1$ units to the right; for $n = 8$, by shifting $15u + 1$ units to the right.

In particular, the AR quiver of the u -cluster category of type E_n , $n = 6, 7, 8$, has the shape of a cylinder, except when $n = 6$ and u is odd where it has the shape of a Möbius band.

3. CLUSTER TILTING OBJECTS

The notion of a u -cluster tilting object in a triangulated category was recalled in Section 2. There is also a definition in abelian categories, cf. [17, sec. 2]. An object X of an abelian category is called u -cluster tilting if

- (i) $\text{Ext}^1(X, t) = \cdots = \text{Ext}^u(X, t) = 0 \Leftrightarrow t \in \text{add } X$,
- (ii) $\text{Ext}^1(t, X) = \cdots = \text{Ext}^u(t, X) = 0 \Leftrightarrow t \in \text{add } X$.

Over selfinjective algebras, there is the following simple connection between u -cluster tilting objects in the module category (which is abelian) and the stable module category (which is triangulated).

Proposition 3.1. *Let A be a selfinjective k -algebra and let X be a u -cluster tilting object of the module category $\text{mod } A$. Then X is also a u -cluster tilting object of the stable module category $\text{stab } A$.*

Proof. Since A is selfinjective, the suspension functor Σ provides us with isomorphisms

$$\underline{\text{Hom}}(M, \Sigma^i N) \cong \text{Ext}^i(M, N)$$

for M and N in $\text{mod } A$ and $i \geq 1$. Here $\underline{\text{Hom}}$ denotes morphisms in $\text{stab } A$.

On one hand, this implies

$$\underline{\mathrm{Hom}}(X, \Sigma^1 X) = \cdots = \underline{\mathrm{Hom}}(X, \Sigma^u X) = 0.$$

On the other hand, suppose that t in $\mathbf{stab} A$ satisfies

$$\underline{\mathrm{Hom}}(X, \Sigma^1 t) = \cdots = \underline{\mathrm{Hom}}(X, \Sigma^u t) = 0.$$

Then

$$\mathrm{Ext}^1(X, t) = \cdots = \mathrm{Ext}^u(X, t) = 0,$$

so t is in $\mathbf{add} X$ viewed in $\mathbf{mod} A$. But then t is clearly also in $\mathbf{add} X$ viewed in $\mathbf{stab} A$.

A similar argument shows that

$$\underline{\mathrm{Hom}}(t, \Sigma^1 X) = \cdots = \underline{\mathrm{Hom}}(t, \Sigma^u X) = 0$$

implies that t is in $\mathbf{add} X$ viewed in $\mathbf{stab} A$. \square

4. DYNKIN TYPE A

4.a. Nakayama algebras. This subsection proves part (i) of Theorem A from the introduction.

For integers $N, n \geq 1$, consider the Nakayama algebra $B_{N,n+1}$ defined as the path algebra of the circular quiver with N vertices and all arrows pointing in the same direction, modulo the ideal generated by paths of length $n+1$.

This is a selfinjective algebra of tree class A_n . The stable AR quiver of $B_{N,n+1}$ has the shape of a cylinder and can be obtained as $\mathbb{Z}A_n$ modulo a shift by N units to the right.

On the other hand, as we saw in Section 2, if u is even then the u -cluster category of type A_n has an AR quiver which can be obtained as $\mathbb{Z}A_n$ modulo a shift by $\frac{u}{2}(n+1) + 1$ units to the right. Indeed, this is no coincidence.

Theorem 4.1. *Let $u \geq 2$ be an even integer and let $n \geq 1$ be an integer. Set*

$$N = \frac{u}{2}(n+1) + 1.$$

Then the u -cluster category of type A_n is equivalent as a triangulated category to the stable module category $\mathbf{stab} B_{N,n+1}$.

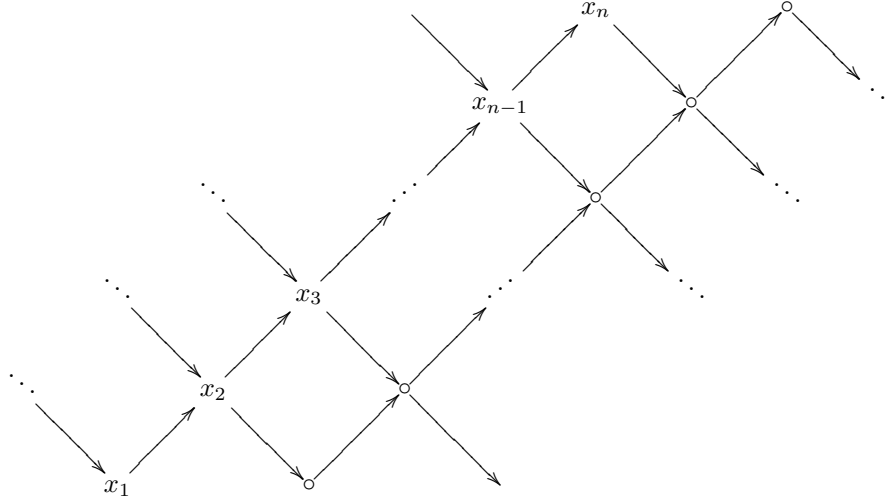
Proof. For $n = 1$ the theorem states that the u -cluster category of type A_1 is triangulated equivalent to $\mathbf{stab} B_{u+1,2}$. This is true by the observation that both categories have AR quiver a disconnected union of $u+1$ vertices, with suspension equal to a cyclic shift by one vertex.

We now assume $n \geq 2$, in which case the relevant categories are connected. By Keller and Reiten's Morita theorem for u -cluster categories [20, thm. 4.2], we need to show three things for the stable module category $\mathbf{stab} B_{N,n+1}$.

- It has *Calabi-Yau dimension* $u+1$.
- It has a *u -cluster tilting object* X with endomorphism ring kA_n .
- The object X has *vanishing of negative self-extensions* in the sense that $\underline{\mathrm{Hom}}(X, \Sigma^{-i} X) = 0$ for $i = 1, \dots, u-1$.

According to this, the proof is divided into three sections. Note the shift in the indices compared to [20]: their d -cluster categories are u -cluster categories for $u = d-1$ in our notation.

Calabi-Yau dimension. We must show that $\mathbf{stab} B_{N,n+1}$ has Calabi-Yau dimension $u+1$, and we can do so using the results by Dugas in [8]. To apply his result from [8, thm. 6.1(2)] in our case of type A_n where $n \geq 2$, we need the Coxeter number $h_{A_n} = n+1$, and we have to observe that

FIGURE 7. The indecomposable modules x_1, \dots, x_n for the Nakayama algebra

in Asashiba's notation from [1, appendix] the Nakayama algebra $B_{N,n+1}$ has the form $(A_n, \frac{N}{n}, 1)$ where $f = \frac{N}{n}$ is the *frequency*. Then [8, thm. 6.1(2)] states that the stable module category $\text{stab } B_{N,n+1}$ has Calabi-Yau dimension $2r + 1$ where $r \equiv -(h_{A_n})^{-1} \pmod{fn}$ and $0 \leq r < fn$. Since $f = \frac{N}{n}$ the value of r is determined by $0 \leq r < N$ and $r \equiv -(h_{A_n})^{-1} \pmod{N} = -(n+1)^{-1} \pmod{N}$. By our assumptions in Theorem 4.1 we have that $u = 2\ell$ is even and that $N = \frac{u}{2}(n+1) + 1 = \ell(n+1) + 1$. Then the condition for the value of r reads $r \equiv -(n+1)^{-1} \pmod{\ell(n+1) + 1}$ which together with $0 \leq r < N = \ell(n+1) + 1$ clearly forces $r = \ell$. Therefore we can deduce that $\text{stab } B_{N,n+1}$ has Calabi-Yau dimension $2r + 1 = 2\ell + 1 = u + 1$, as desired.

u-cluster tilting object. To find a u -cluster tilting object X in $\text{stab } B_{N,n+1}$, by Proposition 3.1 it suffices to find a u -cluster tilting module X in the module category $\text{mod } B_{N,n+1}$. We define X to be the direct sum of the projective indecomposable $B_{N,n+1}$ -modules and the indecomposable modules x_1, \dots, x_n whose position in the stable AR quiver of $B_{N,n+1}$ is given by Figure 7. For the (uniserial) Nakayama algebras $B_{N,n+1}$ it is well-known that the i th layer from the bottom of the stable AR quiver contains precisely the non-projective indecomposable modules of dimension i (see e.g. [3, cor. V.4.2]). Moreover, the arrow from x_i to x_{i+1} in the above picture is a monomorphism for each i . From this follows easily that the stable endomorphism ring of the module X is isomorphic to kA_n .

We now show that the module X defined above is u -cluster tilting. The u -cluster tilting modules (also called maximal u -orthogonal modules) for selfinjective algebras of finite type with tree class A_n were described combinatorially in [17, sec. 4]. We briefly sketch the main ingredients and refer to [17] for details. On the stable AR quiver of $B_{N,n+1}$ one introduces a coordinate system as in Figure 8. The first coordinate has to be taken modulo N . To each vertex x in the stable AR quiver one associates a 'forbidden region' $H^+(x)$ which is just the rectangle spanned from x to the right; more precisely, if $x = (i, j)$, then $H^+(x)$ is the rectangle with corners $x = (i, j)$, $(i, i + n + 1)$, $(j - 2, i + n + 1)$ and $(j - 2, j)$ shown in Figure 9. Define an automorphism ω on the stable AR quiver by setting $\omega(i, j) = (j - n - 2, i + 1)$ and let τ be the usual AR translation, $\tau(i, j) = (i - 1, j - 1)$.

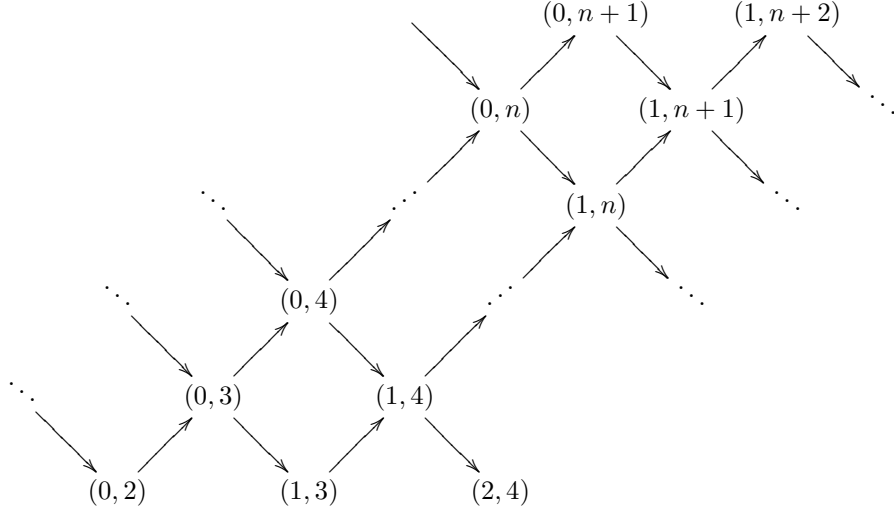
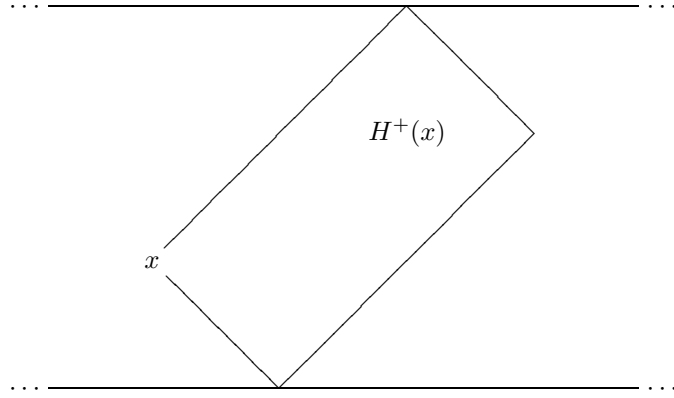
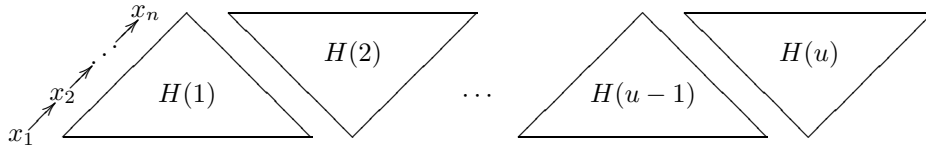


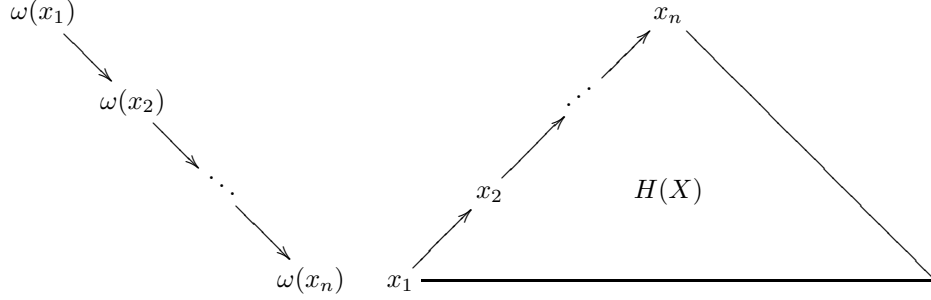
FIGURE 8. The coordinate system for the Nakayama algebra

FIGURE 9. The set $H^+(x)$ in Dynkin type A FIGURE 10. The sets $H(i)$ for the Nakayama algebra

Then a subset S of the vertex set M in the stable AR quiver is called u -cluster tilting if

$$M \setminus S = \bigcup_{x \in S, 0 < i \leq u} H^+(\tau^{-1}\omega^{-i+1}x).$$

For our particular choice of the $B_{N,n+1}$ -module X the set S is given by the above ‘slice’ x_1, \dots, x_n . Then the straightforward, but crucial, observation is that for $i = 1, \dots, u$ the sets $H(i) = \bigcup_{x \in S} H^+(\tau^{-1}\omega^{-i+1}x)$ are as shown in Figure 10. I.e., each $H(i)$ contains all the vertices in a

FIGURE 11. The set $H(X)$ and direct summands of X and ωX for the Nakayama algebra

triangular region of the stable AR quiver with each edge of the triangle containing n vertices.

Recall that u is even by assumption. In total, the union of the forbidden regions $\bigcup_{0 < i \leq u} H(i)$ covers precisely the region of the stable AR quiver between the slice x_1, \dots, x_n and the shift of it by $\frac{u}{2}(n+1)+1$ units to the right. But the stable AR quiver has a circumference of $N = \frac{u}{2}(n+1)+1$ units, so it is clear from the above discussion that the set S is u -cluster tilting and that, accordingly, the $B_{N,n+1}$ -module X is indeed u -cluster tilting.

Vanishing of negative self-extensions. We must show $\underline{\text{Hom}}(X, \Sigma^{-i}X) = 0$ for $i = 1, \dots, u-1$. For this, we can view X in $\text{stab } B_{N,n+1}$ where it has the n indecomposable summands x_1, \dots, x_n . Given non-projective indecomposable $B_{N,n+1}$ -modules v and w , observe that by [17, sec. 4.2 and prop. 4.4.3], we have $\underline{\text{Hom}}(v, w) = 0$ precisely if the vertex of w is outside the forbidden region $H^+(v)$. So we need to check that all vertices corresponding to indecomposable summands of $\Sigma^{-i}X$ for $i = 1, \dots, u-1$ are outside the forbidden region $H(X) = \bigcup_j H^+(x_j)$, where the union is over the indecomposable summands x_j in X .

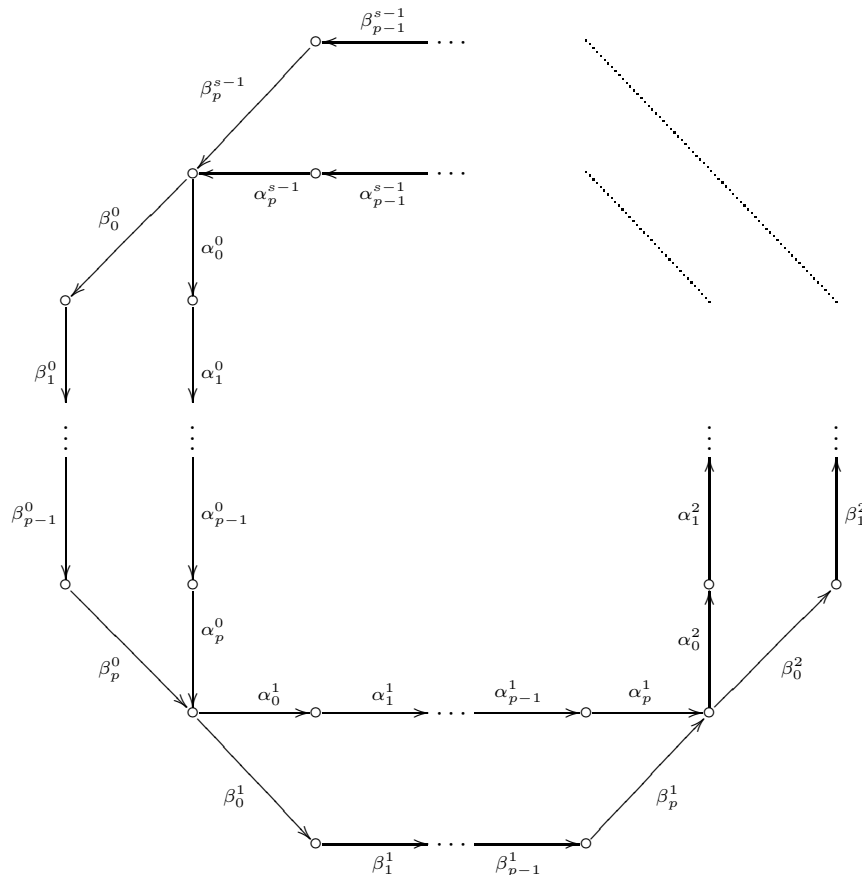
Now, the action of Σ^{-1} on the stable AR quiver is just ω . For instance, Figure 11 shows the forbidden region along with the direct summands of X and of ωX . It is clear that the $\omega(x_j)$ fall outside $H(X)$. More generally, ω moves vertices to the left, so the only way we could fail to get $\underline{\text{Hom}}(X, \Sigma^{-i}X) = 0$ would be if we took i so large that the $\omega^i(x_j)$ made it all the way around the stable AR quiver and reached the forbidden region from the right. Let us check that this does not happen: ω^2 is just a shift by $n+1$ units to the left, and hence $\omega^{u-2} = (\omega^2)^{\frac{u}{2}-1}$ is a shift by $(\frac{u}{2}-1)(n+1) = N - (n+2)$ units to the left. Since the stable AR quiver has circumference N it is clear that by applying ω^{u-1} we do not reach the forbidden region from the right. \square

4.b. Möbius algebras. This subsection proves part (ii) of Theorem A from the introduction.

For integers $p, s \geq 1$, consider the Möbius algebra $M_{p,s}$. Following the notation in [1, app. A2.1.2], this is the path algebra of the quiver shown in Figure 12 modulo the following relations.

- (i) $\alpha_p^i \cdots \alpha_0^i = \beta_p^i \cdots \beta_0^i$ for each $i \in \{0, \dots, s-1\}$.
- (ii) $\beta_0^{i+1} \alpha_p^i = 0, \alpha_0^{i+1} \beta_p^i = 0$ for each $i \in \{0, \dots, s-2\}$,
 $\alpha_0^0 \alpha_p^{s-1} = 0, \beta_0^0 \beta_p^{s-1} = 0$.
- (iii) Paths of length $p+2$ are equal to zero.

This is a selfinjective algebra of tree class A_{2p+1} . In the notation of [1, app. A2.1.2] the Möbius algebra $M_{p,s}$ is of the form $(A_{2p+1}, s, 2)$. The stable AR quiver of $M_{p,s}$ has the shape of a Möbius



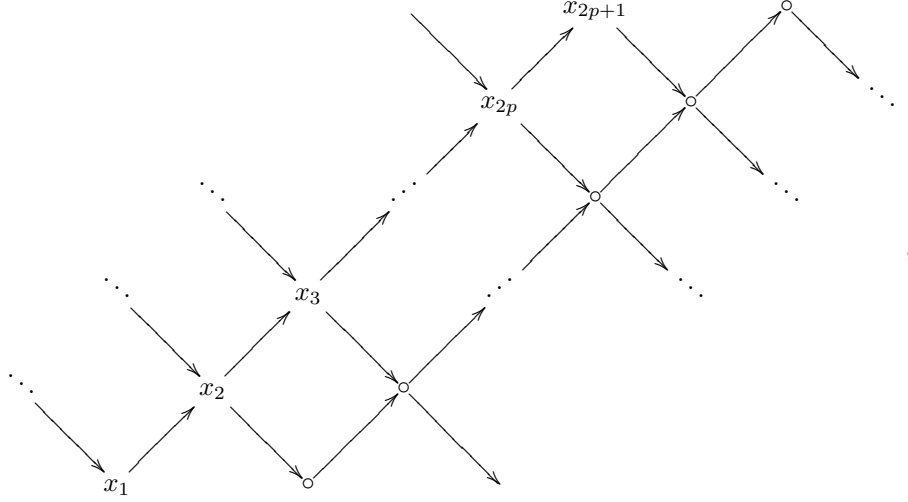
band and can be obtained as $\mathbb{Z}A_{2p+1}$ modulo a reflection in the horizontal centre line composed with a shift by $s(2p+1)$ units to the right, see [24].

Theorem 4.2. *Let $u \geq 1$ be an odd integer and let $p, s \geq 1$ be integers for which*

$$s(2p+1) = u(p+1) + 1.$$

Proof. Like the proof of Theorem 4.1, this proof is divided into three sections verifying the conditions in Keller and Reiten’s Morita theorem [20, thm. 4.2].

$$K_{p,s} = \inf \left\{ r \mid r \geq 1, r(p+1) \equiv 1 \pmod{s}, \text{ and } \frac{r(s+p+1)-1}{s} \text{ is even} \right\}.$$

FIGURE 13. The indecomposable modules x_1, \dots, x_{2p+1} for the Möbius algebra

Let us determine the number $K_{p,s}$ for the values of u, p and s given by the assumptions of the theorem. We have

$$u + 2 = \frac{s(2p+1) - 1}{p+1} + 2 = \frac{(s+1)(2p+1)}{p+1}.$$

Since $\gcd(p+1, 2p+1) = 1$, we deduce that $p+1$ divides $s+1$. Moreover, the integer $\frac{s+1}{p+1}$ is odd since u is odd by assumption. Now, for the condition $r(p+1) \equiv 1 \pmod{s}$, the integer $\frac{s+1}{p+1}$ is clearly the minimal (positive) solution. Moreover, for this value $r = \frac{s+1}{p+1}$ we have that

$$\frac{r(s+p+1) - 1}{s} = \frac{(s+1)(s+p+1) - (p+1)}{(p+1)s} = \frac{s+p+2}{p+1} = \frac{s+1}{p+1} + 1$$

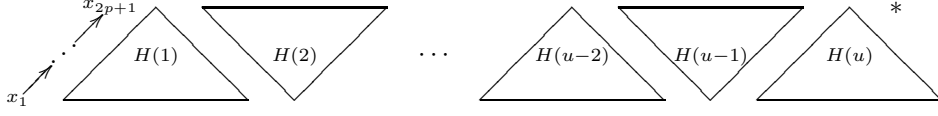
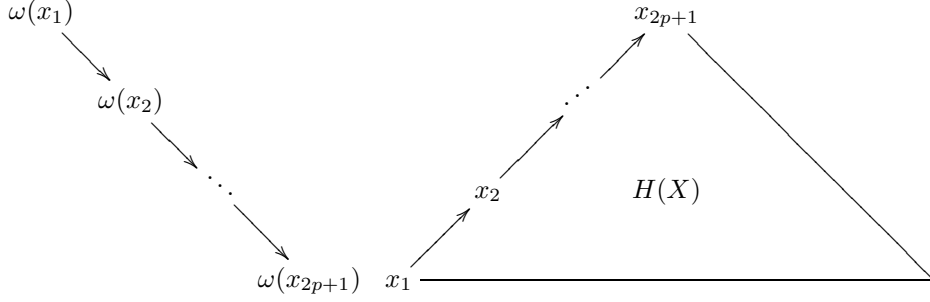
is even. Hence $K_{p,s} = \frac{s+1}{p+1}$, and we conclude that $\text{stab } M_{p,s}$ has Calabi-Yau dimension

$$\begin{aligned} K_{p,s}(2p+1) - 1 &= \frac{(s+1)(2p+1)}{p+1} - 1 = \frac{(s+1)(2p+1) - 2(p+1)}{p+1} + 1 \\ &= \frac{s(2p+1) - 1}{p+1} + 1 = u + 1, \end{aligned}$$

where the last equality holds by assumption on u .

u-cluster tilting object. To find a u -cluster tilting object X in $\text{stab } M_{p,s}$, recall that the projective indecomposable $M_{p,s}$ -modules are either uniserial or biserial, and that correspondingly, the vertices in the quiver of $M_{p,s}$ are called uniserial or biserial. The position of the corresponding simple modules in the stable AR quiver is well-known; in particular, the simple modules corresponding to biserial vertices occur in the centre line of the stable AR quiver. As in the case of Nakayama algebras, we define the module X as the direct sum of the projective indecomposable modules and the indecomposable modules x_1, \dots, x_{2p+1} lying on a slice as in Figure 13 such that the module x_{p+1} is a simple module S_v corresponding to a biserial vertex v of the quiver of $M_{p,s}$. The other modules in this slice can also be described. For $j = 1, \dots, p$, the module x_j is the uniserial module of length $p+2-j$ with top S_v , and the module x_{p+j+1} is the uniserial module of length $j+1$ with socle S_v .

In particular, the bottom p maps are epimorphisms and the upper p maps are monomorphisms. The composition of all $2p$ maps in such a slice is non-zero, mapping the top onto the socle. Most

FIGURE 14. The sets $H(i)$ for the Möbius algebraFIGURE 15. The set $H(X)$ and direct summands of X and ωX for the Möbius algebra

importantly for us, it does not factor through a projective module, i.e., it is a non-zero morphism in the stable module category of $M_{p,s}$. From this it follows easily that the stable endomorphism ring of the module X is isomorphic to kA_{2p+1} .

We now show that X is u -cluster tilting. This argument is also analogous to the Nakayama algebra case. The crucial difference is that now u is odd. Hence, the forbidden regions defined in [17] and discussed in the proof of Theorem 4.1 are as in Figure 14. The method from the proof of Theorem 4.1 shows that, to see that X is u -cluster tilting, it is sufficient to see that the vertex x_1 is identified with the vertex $*$. However, each $H(i)$ contains the vertices of the stable AR quiver in an equilateral triangular region with edges having $2p+1$ vertices. So in order for x_1 to be identified with $*$, we must identify after $\frac{u-1}{2}(2p+2) + (p+2) = u(p+1) + 1$ units. But in fact, one gets the stable AR quiver of $M_{p,s}$ from $\mathbb{Z}A_{2p+1}$ by identifying after $s(2p+1)$ units, and by the assumption of the theorem we do indeed have $s(2p+1) = u(p+1) + 1$.

Vanishing of negative self-extensions. We must show $\underline{\text{Hom}}(X, \Sigma^{-i}X) = 0$ for $i = 1, \dots, u-1$. The proof is analogous to the Nakayama case: The action of Σ^{-1} on the stable AR quiver is again just ω , and the forbidden region of X along with the direct summands of X and of ωX are as in Figure 15. The only way we could fail to get $\underline{\text{Hom}}(X, \Sigma^{-i}X) = 0$ would be if we took i so large that the $\omega^i(x_j)$ made it all the way around the stable AR quiver and reached the forbidden region from the right. In fact, let us look at the largest relevant integer, $u-1$. As ω^2 is just a shift by $2p+2$ units to the left, we have that $\omega^{u-1} = (\omega^2)^{\frac{u-1}{2}}$ is a shift by $\frac{u-1}{2}(2p+2) = (u-1)(p+1) = s(2p+1) - (p+2)$ units to the left. The stable AR quiver has a circumference of $s(2p+1)$ units, so the $\omega^{u-1}(x_j)$ lie strictly to the right of the forbidden region. (Note that the stable AR quiver is a Möbius band, and the change of orientation means that, although $u-1$ is even, the $\omega^{u-1}(x_j)$ form a diagonal line perpendicular, not parallel, to the line of the x_j .) So $\underline{\text{Hom}}(X, \Sigma^{-i}X)$ is zero for $i = u-1$, and hence certainly also for all values $i = 1, \dots, u-1$. This completes the proof. \square

Remark 4.3. Note that as a special case of Theorem 4.2, the 1-cluster category of type A_3 is triangulated equivalent to $\text{stab } M_{1,1}$. The Möbius algebra $M_{1,1}$ is isomorphic to the preprojective algebra of Dynkin type A_3 .

This is the only case where a 1-cluster category is triangulated equivalent to the stable module category of a selfinjective algebra of finite representation type and tree class A_n . This follows from the complete classification of representation-finite selfinjective algebras of stable Calabi-Yau dimension 2 given in [9, cor. 3.10].

5. DYNKIN TYPE D

This section proves Theorem D from the introduction.

Asashiba's paper [2] gives a derived and stable equivalence classification of selfinjective algebras of finite representation type. If the tree class of the stable AR quiver is Dynkin type D , then there are three families of representatives of algebras denoted

- $(D_n, s, 1)$ with $n \geq 4$, $s \geq 1$,
- $(D_n, s, 2)$ with $n \geq 4$, $s \geq 1$,
- $(D_{3m}, \frac{s}{3}, 1)$ with $m \geq 2$, $s \geq 1$, $3 \nmid s$.

It follows from [4, cor. 1.7] that the stable AR quivers of these algebras are cylinders with the following circumferences.

- For $(D_n, s, 1)$ and $(D_n, s, 2)$ the circumference is $s(2n - 3)$.
- For $(D_{3m}, \frac{s}{3}, 1)$ the circumference is $s(2m - 1)$.

By Subsection 2.b, the AR quiver of the u -cluster category of type D_n is a cylinder of circumference $u(n - 1) + 1$. So in order for the stable categories $\text{stab}(D_n, s, 1)$ or $\text{stab}(D_n, s, 2)$ to be u -cluster categories we need

$$u(n - 1) + 1 = s(2n - 3).$$

In particular, this implies

$$u \equiv -(n - 1)^{-1} \equiv -2 \pmod{2n - 3}.$$

Likewise, for the stable category $\text{stab}(D_{3m}, \frac{s}{3}, 1)$ to be a u -cluster category, we need

$$u(3m - 1) + 1 = s(2m - 1). \tag{1}$$

In particular, this implies

$$u \equiv -m^{-1} \equiv -2 \pmod{2m - 1}.$$

Moreover, recall that in the definition of the algebras $(D_{3m}, \frac{s}{3}, 1)$ the case $3 \mid s$ is excluded. In the situation of equation (1) we have

$$3 \nmid s \iff u(3m - 1) + 1 \not\equiv 0 \pmod{3(2m - 1)} \iff u \not\equiv -(3m - 1)^{-1} \equiv -2 \pmod{6m - 3}.$$

Indeed, these conditions turn out also to be sufficient. Note that, setting $n = 3m$, the forbidden case $u \equiv -2 \pmod{6m - 3}$ for the algebras $(D_{3m}, \frac{s}{3}, 1)$ is precisely the case $u \equiv -2 \pmod{2n - 3}$ in which the algebras $(D_n, s, 1)$ and $(D_n, s, 2)$ can be applied.

The main result of this section is the following which restates Theorem D from the introduction.

Theorem 5.1. *Let m, n, u be integers with $u \geq 1$.*

- (i) *Suppose that $n \geq 4$ is even and $u \equiv -2 \pmod{2n - 3}$. Then the u -cluster category of type D_n is equivalent as a triangulated category to the stable module category*

$$\text{stab}(D_n, \frac{u(n - 1) + 1}{2n - 3}, 1).$$

(ii) Suppose that $n \geq 5$ is odd and $u \equiv -2 \pmod{2n-3}$.

If u is even, then the u -cluster category of type D_n is triangulated equivalent to the stable module category $\mathbf{stab}(D_n, \frac{u(n-1)+1}{2n-3}, 1)$.

If u is odd, then the u -cluster category of type D_n is triangulated equivalent to the stable module category $\mathbf{stab}(D_n, \frac{u(n-1)+1}{2n-3}, 2)$.

(iii) Suppose that $m \geq 2$ and $u \equiv -2 \pmod{2m-1}$ but $u \not\equiv -2 \pmod{6m-3}$. Suppose moreover that not both m and u are odd. Then the u -cluster category of type D_{3m} is equivalent as a triangulated category to the stable module category $\mathbf{stab}(D_{3m}, \frac{s}{3}, 1)$ where $s = \frac{u(3m-1)+1}{2m-1}$.

Proof. As in type A, the proof is divided into three sections verifying the conditions in Keller and Reiten's Morita theorem [20, thm. 4.2].

Calabi-Yau dimension. We must show that each of the stable module categories occurring in the theorem has Calabi-Yau dimension $u+1$.

For part (i) we suppose that $n \geq 4$ is even and we consider the algebra $(D_n, \frac{u(n-1)+1}{2n-3}, 1)$. The Calabi-Yau dimension of its stable module category can be determined using [8, thm. 6.1], in which both parts can apply. The relevant invariants occurring there are the frequency $f = \frac{u(n-1)+1}{2n-3}$, the Coxeter number $h_{D_n} = 2n-2$ and the related number $h_{D_n}^* = h_{D_n}/2 = n-1$, and $m_{D_n} = h_{D_n} - 1 = 2n-3$.

If [8, thm. 6.1(1)] applies then the Calabi-Yau dimension d of $\mathbf{stab}(D_n, \frac{u(n-1)+1}{2n-3}, 1)$ satisfies

$$d \equiv 1 - (h_{D_n}^*)^{-1} \pmod{f m_{D_n}} \equiv 1 - (n-1)^{-1} \pmod{u(n-1)+1}$$

and $0 < d \leq u(n-1)+1$. Upon multiplication with $n-1$ this becomes $d(n-1) \equiv n-2 \pmod{u(n-1)+1}$ which is easily checked to be satisfied by $d = u+1$.

If [8, thm. 6.1(2)] applies then the Calabi-Yau dimension d of $\mathbf{stab}(D_n, \frac{u(n-1)+1}{2n-3}, 1)$ has the form $d = 2r+1$ where r is determined by

$$r \equiv -(h_{D_n})^{-1} \pmod{f m_{D_n}} \equiv -(2n-2)^{-1} \pmod{u(n-1)+1} \quad (2)$$

and $0 \leq r < u(n-1)+1$. Since [8, thm. 6.1(2)] applies we know from the assumptions stated in [8, thm. 6.1(1)] that $2 \nmid f = \frac{u(n-1)+1}{2n-3}$ from which it follows that u is even (since n is even). Setting $r = \frac{u}{2}$ it is readily checked that it satisfies (2). Therefore the Calabi-Yau dimension is $2r+1 = u+1$, as required.

For part (ii), we suppose that $n \geq 5$ is odd and we consider the algebras $(D_n, \frac{u(n-1)+1}{2n-3}, 1)$ and $(D_n, \frac{u(n-1)+1}{2n-3}, 2)$, depending on whether u is even or odd.

If u is even then the Calabi-Yau dimension of $\mathbf{stab}(D_n, \frac{u(n-1)+1}{2n-3}, 1)$ can be determined using [8, thm. 6.1(2)] (note that [8, thm. 6.1(1)] only applies for n even). The only difference to the case of n even is the invariant $h_{D_n}^*$ which is now equal to $2n-2$ instead of $n-1$. But this invariant does not occur in [8, thm. 6.1(2)] so the proof for n even carries over verbatim and gives that $\mathbf{stab}(D_n, \frac{u(n-1)+1}{2n-3}, 1)$ has Calabi-Yau dimension $u+1$.

If u is odd (and $n \geq 5$ is still odd) then the Calabi-Yau dimension of $\mathbf{stab}(D_n, \frac{u(n-1)+1}{2n-3}, 2)$ can be determined using [8, prop. 7.3]. Note that since n is odd, the frequency $f = \frac{u(n-1)+1}{2n-3}$ is odd as well, and hence [8, prop. 7.3(1)] applies. From this we get that the Calabi-Yau dimension of $\mathbf{stab}(D_n, \frac{u(n-1)+1}{2n-3}, 2)$ is of the form $d = 2r$ where $r \equiv (n-2)(2n-2)^{-1} \pmod{u(n-1)+1}$ and $0 < r < u(n-1)+1$. Upon multiplication with $2n-2$ the latter equation becomes $2r(n-1) \equiv$

$n - 2 \bmod (u(n - 1) + 1)$ which is easily seen to be satisfied by $r = \frac{u+1}{2}$. Therefore, the Calabi-Yau dimension is $d = 2r = u + 1$, as required.

For part (iii), we consider the algebras $(D_{3m}, \frac{s}{3}, 1)$ where $s = \frac{u(3m-1)+1}{2m-1}$. The Calabi-Yau dimension of the stable module category can again be determined using [8, thm. 6.1].

If m is even then the invariants we need are the frequency $f = \frac{s}{3} = \frac{u(3m-1)+1}{3(2m-1)}$, the Coxeter number $h_{D_{3m}} = 6m - 2$ and the related numbers $m_{D_{3m}} = h_{D_{3m}} - 1 = 6m - 3$, and $h_{D_{3m}}^* = h_{D_{3m}}/2 = 3m - 1$.

If [8, thm. 6.1(1)] applies then the Calabi-Yau dimension d of $\mathbf{stab}(D_{3m}, \frac{s}{3}, 1)$ is determined by

$$d \equiv 1 - (h_{D_{3m}}^*)^{-1} \bmod f m_{D_{3m}} \equiv 1 - (3m - 1)^{-1} \bmod (u(3m - 1) + 1)$$

and $0 < d \leq u(3m - 1) + 1$. Clearly, $d = u + 1$ satisfies these properties and hence the Calabi-Yau dimension is $u + 1$, as claimed.

If [8, thm. 6.1(2)] applies then the Calabi-Yau dimension d of $\mathbf{stab}(D_{3m}, \frac{s}{3}, 1)$ is of the form $d = 2r + 1$ where r is determined by

$$r \equiv -(h_{D_{3m}})^{-1} \bmod f m_{D_{3m}} \equiv -(6m - 2)^{-1} \bmod (u(3m - 1) + 1)$$

and $0 \leq r < u(3m - 1) + 1$. Note that our assumptions in this case imply that u is even; otherwise the frequency f would be even and we would be in the situation of [8, thm. 6.1(1)]. Setting $r = \frac{u}{2}$ is easily seen to satisfy the above properties, i.e. the Calabi-Yau dimension is $2r + 1 = u + 1$, as desired.

Finally, if m is odd then only [8, thm. 6.1(2)] can apply. The computation of the Calabi-Yau dimension carries over verbatim from the previous one; in fact, by assumption in part (iii) of our theorem u has to be even (since m is odd). Hence also for m odd and u even we get the Calabi-Yau dimension of $\mathbf{stab}(D_{3m}, \frac{s}{3}, 1)$ to be $u + 1$, as required.

u-cluster tilting object. To find a u -cluster tilting object X in the stable module category, the method is the same for parts (i)–(iii). In part (iii) we set $n = 3m$ so that in each case n denotes the number of vertices in the underlying Dynkin quiver of type D_n .

Let X be the sum of the projective indecomposable modules and the indecomposable modules $x_1, \dots, x_{n-2}, x_{n-1}^-, x_{n-1}^+$ whose positions in the stable AR quiver of the relevant algebra are given by Figure 16. We show that X is u -cluster tilting in the stable module category. By Proposition 3.1, it is enough to prove that it is u -cluster tilting in the abelian category of modules. Following [17, def. 4.2], introduce a coordinate system on the stable AR quiver as in Figure 17. To each vertex x in the stable AR quiver, associate a ‘forbidden region’ $H^+(x)$ defined as in Figure 18 (see [17, sec. 4.2]), with the proviso that if x is not one of the ‘exceptional’ vertices indicated by superscripts $+$ and $-$, then $H^+(x)$ contains all the exceptional vertices along the relevant part of the top line in the diagram, but if x is exceptional, say $x = (i, i + n)^+$, then $H^+(x)$ only contains half the exceptional vertices along the relevant part of the top line, namely $(i, i + n)^+, (i + 1, i + n + 1)^-, (i + 2, i + n + 2)^+, \dots$, starting with x itself. Define the following automorphisms of the stable AR quiver: θ is the identity on the non-exceptional vertices and switches $(i, i + n)^+$ and $(i, i + n)^-$. The AR translation τ is given by moving each vertex one unit to the left. And finally, $\omega = \theta(\tau\theta)^{n-1}$. A subset S of the vertex set M in the stable AR quiver is called u -cluster tilting if

$$M \setminus S = \bigcup_{x \in S, 0 < i \leq u} H^+(\tau^{-1}\omega^{-i+1}x),$$

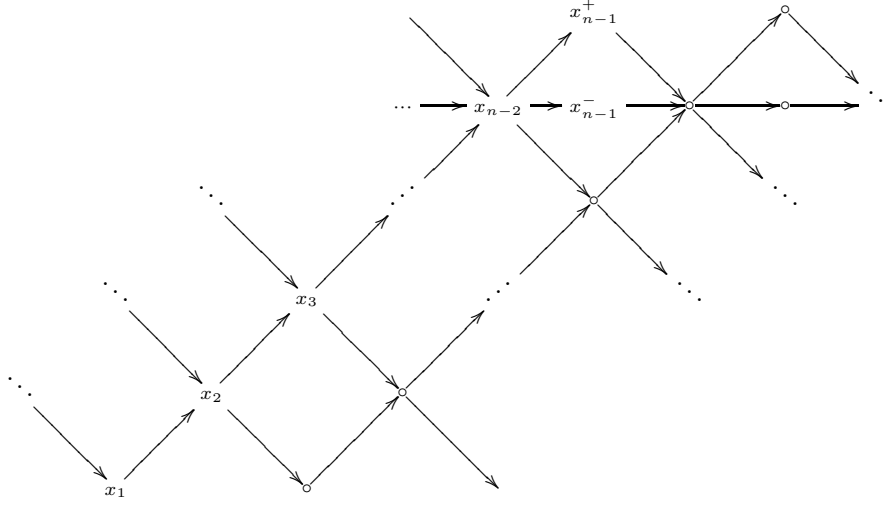


FIGURE 16. The indecomposable modules $x_1, \dots, x_{n-2}, x_{n-1}^-, x_{n-1}^+$ in Dynkin type D

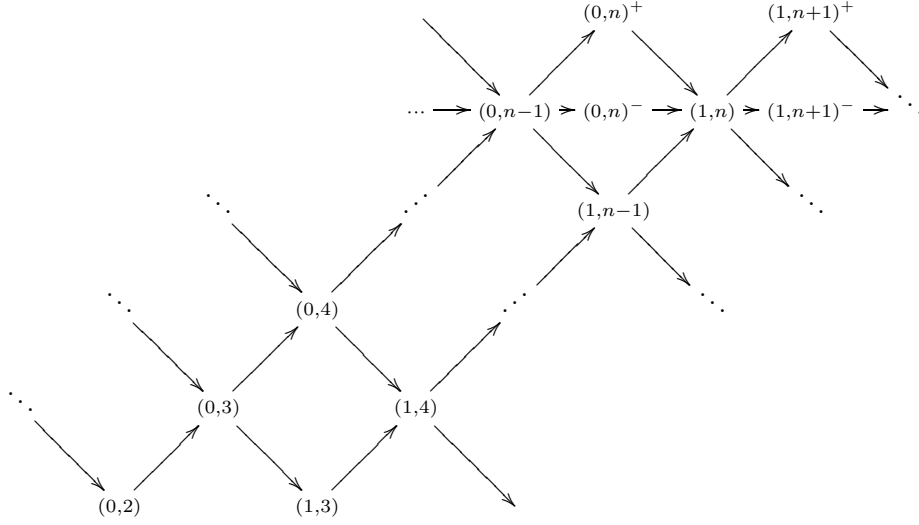


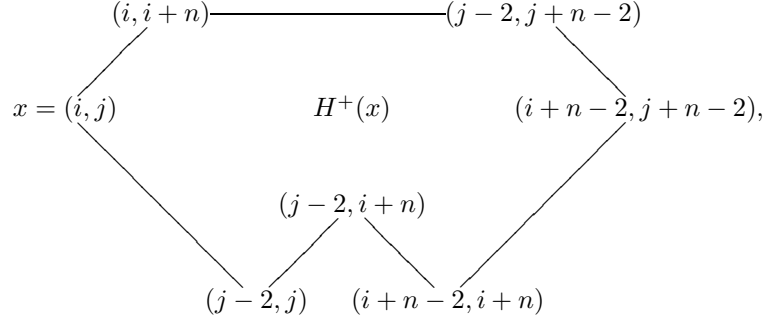
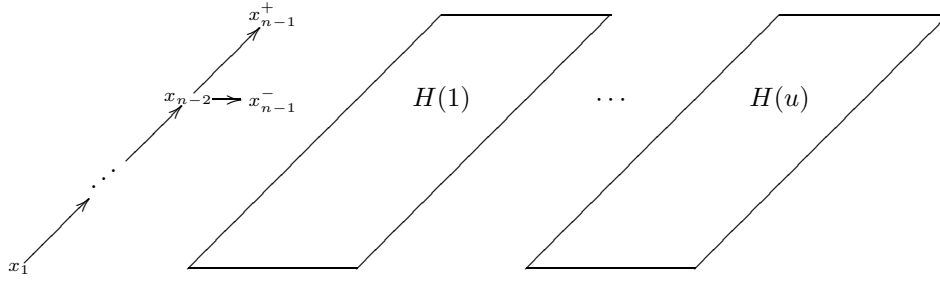
FIGURE 17. The coordinate system in Dynkin type D

see [17, sec. 4.2]. For our choice of X , the set S is given by the modules $x_1, \dots, x_{n-2}, x_{n-1}^-, x_{n-1}^+$. But then the sets

$$H(i) = \bigcup_{x \in S} H^+(\tau^{-1}\omega^{-i+1}x)$$

can easily be verified to sit in the stable AR quiver as in Figure 19 where each parallelogram has $n-1$ vertices on each edge. In total, the union

$$\bigcup_{0 < i \leq u} H(i) = \bigcup_{x \in S, 0 < i \leq u} H^+(\tau^{-1}\omega^{-i+1}x)$$

FIGURE 18. The set $H^+(x)$ in Dynkin type D FIGURE 19. The sets $H(i)$ in Dynkin type D

is a parallelogram with $u(n-1)$ vertices on each horizontal edge. This means that the parallelogram covers precisely the region between the x 's and their shift by $u(n-1) + 1$ units to the right.

By Subsection 2.b, this is exactly the number of units after which $\mathbb{Z}D_n$ is identified with itself to get the stable AR quiver. It follows that S is a u -cluster tilting set of vertices of the stable AR quiver, and hence X is u -cluster tilting in the module category by [17, thm. 4.2.2].

To show that the stable endomorphism algebra $\underline{\text{End}}(X)$ is kD_n , we need to see that for each pair of indecomposable summands x_i and x_j of X , the stable Hom-space $\underline{\text{Hom}}(x_i, x_j)$ is one-dimensional if x_i is below x_j in the stable AR quiver, and zero otherwise.

The self-injective algebras in the theorem in question are standard, so each morphism between indecomposable modules in the stable category is a sum of compositions of sequences of irreducible morphisms between indecomposable modules. Consider such a sequence which composes to a morphism $x_i \rightarrow x_j$.

If, along the sequence, there is an indecomposable y which is not a summand of X , then $y \rightarrow x_j$ factors through a direct sum of indecomposable summands of τX by [3, lem. VIII.5.4]. But then $x_i \rightarrow y \rightarrow x_j$ factors in the same way, and this means that it is zero because $\underline{\text{Hom}}(X, \tau X) = 0$ by the methods used in the proof that X is u -cluster tilting. Hence $x_i \rightarrow x_j$ can be taken to be a sum of compositions of sequences of irreducible morphisms which only pass through indecomposable summands of X . In the stable AR quiver, the arrows between these summands all point upwards, so it follows that $\underline{\text{Hom}}(x_i, x_j)$ is zero unless x_i is below x_j in the stable AR quiver.

On the other hand, if x_i is below x_j , then $\underline{\text{Hom}}(x_i, x_j)$ is non-zero by [17, sec. 4.2 and prop. 4.4.3]. Finally, it follows from [23, satz 3.5] that the dimension of $\underline{\text{Hom}}(x_i, x_j)$ is at most one.

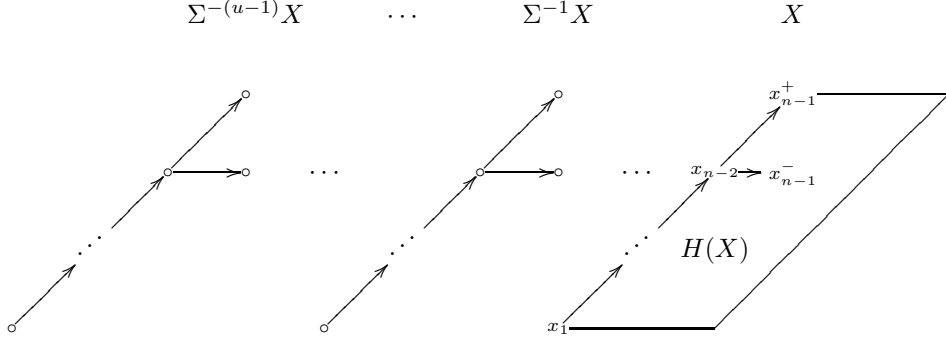


FIGURE 20. The set $H(X)$ and direct summands of $\Sigma^{-(u-1)}X, \dots, \Sigma^{-1}X, X$ in Dynkin type D

Vanishing of negative self-extensions. We must show $\underline{\text{Hom}}(X, \Sigma^{-i}X) = 0$ for $i = 1, \dots, u-1$. If v and w are indecomposable non-projective modules, we have $\underline{\text{Hom}}(v, w) = 0$ precisely if the vertex of w is outside the region $H^+(v)$, see [17, sec. 4.2 and prop. 4.4.3]. So we need to check that all vertices corresponding to indecomposable summands of $\Sigma^{-i}X$ for $i = 1, \dots, u-1$ are outside the forbidden region $H(X) = \bigcup_x H^+(x)$, where the union is over the indecomposable summands of X .

But the action of Σ^{-1} on the stable AR quiver is just ω . So Figure 20 shows the forbidden region along with the $\Sigma^{-i}X$. The only way we could fail to get $\underline{\text{Hom}}(X, \Sigma^{-i}X) = 0$ would be if we took i so large that $\Sigma^i X$ made it all the way around the stable AR quiver and reached the forbidden region from the right.

However, this does not happen: ω , and hence Σ^{-1} , is a move by $n-1$ units to the left, so $\Sigma^{-(u-1)}X$ is moved $(u-1)(n-1)$ units to the left. On the other hand, to reach $H(X)$, one has to move by the circumference of the stable AR quiver minus the horizontal length of $H(X)$ plus one, and this is $u(n-1) + 1 - (n-1) + 1 = (u-1)(n-1) + 2$. \square

Remark 5.2. We would like to stress that in part (iii) of Theorem 5.1, the assumption that at least one of m and u is even is necessary. This assumption was unfortunately missing in our earlier preprint [16]. We are grateful to Alex Dugas for pointing this out to us.

If both m and u are odd then the Calabi-Yau dimension of the stable category $\text{stab}(D_{3m}, \frac{s}{3}, 1)$ cannot be of the form $u+1$, as would be needed for being a u -cluster category. In fact, the Calabi-Yau dimension can again be computed using [8, thm. 6.1(2)] ([8, thm. 6.1(1)] does not apply since m is odd). In particular, the Calabi-Yau dimension is of the form $d = 2r+1$ and hence an odd number which makes it impossible to be equal to $u+1$ (since u is odd).

This happens despite the fact that, for m and u odd, the stable module category $\text{stab}(D_{3m}, \frac{s}{3}, 1)$ and the u -cluster category of type D_{3m} both have as AR quiver a cylinder of circumference $s(2m-1)$. The reason is that under the AR translation τ , the exceptional vertices form a single orbit in the u -cluster category but two orbits in the stable category.

As an explicit example, consider the case when $m = 3$ and $u = 3$. Then the Calabi-Yau dimension of the stable module category $\text{stab}(D_9, \frac{5}{3}, 1)$ is, according to [8, thm. 6.1(2)], of the form $2r+1$ where r is determined by $r \equiv -16^{-1} \pmod{25} \equiv 14 \pmod{25}$ and $0 \leq r < 25$. Thus, $\text{stab}(D_9, \frac{5}{3}, 1)$ has Calabi-Yau dimension 29, which is far from the Calabi-Yau dimension 4 of the 3-cluster category of type D_9 .

Example 5.3. We illustrate our realizability results in type D_n from Theorem 5.1 by considering the situation for some small values of n .

Let us first consider type D_4 . Then parts (ii) and (iii) of Theorem 5.1 do not apply. From part (i) we get for every $u \equiv 3 \pmod{5}$ that the u -cluster category of type D_4 is triangulated equivalent to $\mathbf{stab}(D_4, \frac{3u+1}{5}, 1)$.

Let us now consider type D_6 . From part (i) of Theorem 5.1 we get for every $u \equiv 7 \pmod{9}$ that the u -cluster category of type D_6 is triangulated equivalent to the category $\mathbf{stab}(D_6, \frac{5u+1}{9}, 1)$.

Moreover, from part (iii) of Theorem 5.1 we also get that for every $u \equiv 1 \pmod{9}$ and every $u \equiv 4 \pmod{9}$ that the u -cluster category of type D_6 is triangulated equivalent to $\mathbf{stab}(D_6, \frac{5u+1}{9}, 1)$. Hence, for all $u \equiv 1 \pmod{3}$, we get the u -cluster category of type D_6 as stable module category of a selfinjective algebra.

We remark that the smallest case $u = 1$ states that the 1-cluster category of type D_6 is triangulated equivalent to the stable module category of the preprojective algebra of type A_4 . In fact, the algebra $(D_6, \frac{2}{3}, 1)$ is just this preprojective algebra. This can be considered as the cluster category version of the statement that the preprojective algebra of type A_4 is of cluster type D_6 [13, sec. 19.2]. For more details on the close connection between preprojective algebras and cluster theory we refer to [12].

6. DYNKIN TYPE E

This section proves Theorem E from the introduction.

Asashiba's paper [2] gives that if the tree class of the stable AR quiver is Dynkin type E , then there are four families of representatives of self-injective algebras denoted

- $(E_6, s, 1)$,
- $(E_6, s, 2)$,
- $(E_7, s, 1)$,
- $(E_8, s, 1)$,

all with $s \geq 1$. Recall that in type E , nonstandard algebras do not occur. It follows from [4, cor. 1.7] that the stable AR quivers of these algebras are cylinders with the following circumferences.

- For $(E_6, s, 1)$ and $(E_6, s, 2)$ the circumference is $11s$.
- For $(E_7, s, 1)$ the circumference is $17s$.
- For $(E_8, s, 1)$ the circumference is $29s$.

By Subsection 2.c, the AR quiver of the u -cluster category of type E_6 is a cylinder or a Möbius band of circumference $6u + 1$ (this number is independent of u being even or odd). So in order for the stable categories $\mathbf{stab}(E_6, s, 1)$ or $\mathbf{stab}(E_6, s, 2)$ to be u -cluster categories we need $6u + 1 = 11s$. In particular, this implies

$$u \equiv -6^{-1} \equiv -2 \pmod{11}.$$

Likewise, the AR quiver of the u -cluster category of type E_7 is a cylinder of circumference $9u + 1$. So in order for the stable category $\mathbf{stab}(E_7, s, 1)$ to be a u -cluster category we need $9u + 1 = 17s$. In particular, this implies

$$u \equiv -9^{-1} \equiv -2 \pmod{17}.$$

Finally, the AR quiver of the u -cluster category of type E_8 is a cylinder of circumference $15u + 1$. So in order for the stable module category $\mathbf{stab}(E_8, s, 1)$ to be u -cluster category we need $15u + 1 = 29s$. In particular, this implies

$$u \equiv -15^{-1} \equiv -2 \pmod{29}.$$

Indeed, these conditions turn out also to be sufficient. The main result of this section is the following which restates Theorem E from the introduction.

Theorem 6.1. *Let $u \geq 1$ be an integer.*

- (i) *If $u \equiv -2 \pmod{11}$ then the u -cluster category of Dynkin type E_6 is equivalent as a triangulated category to the stable module category $\text{stab}(E_6, \frac{6u+1}{11}, 1)$ if u is even, and to the stable module category $\text{stab}(E_6, \frac{6u+1}{11}, 2)$ if u is odd.*
- (ii) *If $u \equiv -2 \pmod{17}$ then the u -cluster category of Dynkin type E_7 is equivalent as a triangulated category to the stable module category $\text{stab}(E_7, \frac{9u+1}{17}, 1)$.*
- (iii) *If $u \equiv -2 \pmod{29}$ then the u -cluster category of Dynkin type E_8 is equivalent as a triangulated category to the stable module category $\text{stab}(E_8, \frac{15u+1}{29}, 1)$.*

Proof. As in types A and D, the proof is divided into three sections verifying the conditions in Keller and Reiten's Morita theorem [20, thm. 4.2].

Calabi-Yau dimension. We must show that the relevant stable module categories have Calabi-Yau dimension $u + 1$, and again we do so using the results by Dugas from [8].

First, consider the algebras $(E_6, \frac{6u+1}{11}, 1)$; in particular u is assumed to be even. Then [8, thm. 6.1(2)] applies. Note that the invariants occurring there for type E_6 are given by: The frequency $f = \frac{6u+1}{11}$, the Coxeter number $h_{E_6} = 12$, and $m_{E_6} = h_{E_6} - 1 = 11$. The Calabi-Yau dimension of $\text{stab}(E_6, \frac{6u+1}{11}, 1)$ is then of the form $2r + 1$ where

$$r \equiv -(h_{E_6})^{-1} \pmod{f m_{E_6}} = -12^{-1} \pmod{6u+1}$$

and $0 \leq r < 6u + 1$. Since u is even by assumption we can consider the integer $r = \frac{u}{2}$; this clearly satisfies $12r = 6u \equiv -1 \pmod{6u+1}$, and $0 \leq r < 6u + 1$. Therefore the Calabi-Yau dimension of $\text{stab}(E_6, \frac{6u+1}{11}, 1)$ is $2r + 1 = u + 1$, as desired.

Secondly, consider the algebras $(E_6, \frac{6u+1}{11}, 2)$; in particular u is assumed to be odd. Then we can apply [8, prop. 7.4(1)]. The Calabi-Yau dimension of $\text{stab}(E_6, \frac{6u+1}{11}, 2)$ is then equal to $2r$ where $r \equiv 5 \cdot 12^{-1} \pmod{6u+1}$ and $0 < r < 6u + 1$. Setting $r = \frac{u+1}{2}$ (recall that u is odd by assumption) we immediately get that $12r = 6(u+1) \equiv 5 \pmod{6u+1}$ and hence the Calabi-Yau dimension is $2r = u + 1$, as desired.

Thirdly, consider the algebras $(E_7, \frac{9u+1}{17}, 1)$. The Calabi-Yau dimension can again be determined by [8, thm. 6.1]. The relevant invariants for type E_7 are given by: The frequency $f = \frac{9u+1}{17}$, the Coxeter number $h_{E_7} = 18$ with its variant $h_{E_7}^* = h_{E_7}/2 = 9$, and $m_{E_7} = h_{E_7} - 1 = 17$. Note that for type E_7 both parts of [8, thm. 6.1] can possibly apply; we shall show that in either case we get $u + 1$ as Calabi-Yau dimension of the stable module category.

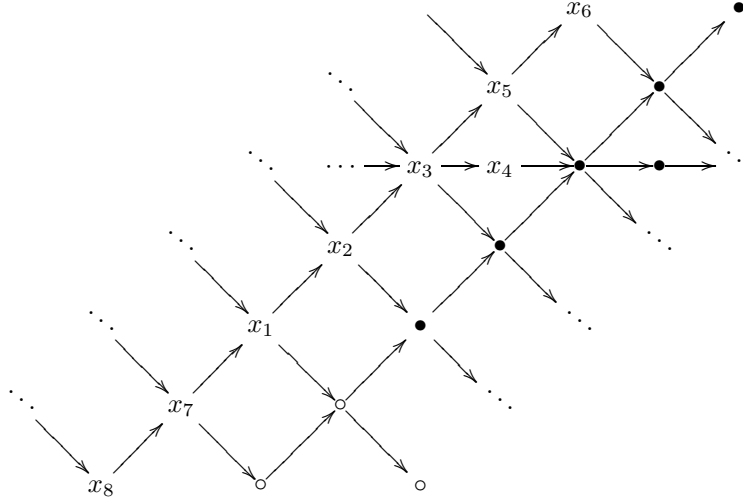
In [8, thm. 6.1(1)] the Calabi-Yau dimension d satisfies

$$d \equiv 1 - (h_{E_7}^*)^{-1} \pmod{f m_{E_7}} = 1 - 9^{-1} \pmod{9u+1}$$

as well as $0 < d \leq 9u + 1$. Clearly, $d = u + 1$ has these properties, and hence the Calabi-Yau dimension is $u + 1$, as desired.

In [8, thm. 6.1(2)] the Calabi-Yau dimension d has the form $d = 2r + 1$ where $r \equiv -18^{-1} \pmod{9u+1}$ and $0 \leq r < 9u + 1$. Note that when [8, thm. 6.1(2)] applies then $2 \nmid f = \frac{9u+1}{17}$ which implies that u is even. Setting $r = \frac{u}{2}$ we immediately see that $18r \equiv 9u \equiv -1 \pmod{9u+1}$, i.e. the Calabi-Yau dimension of the stable category in this case is also $2r + 1 = u + 1$, as desired.

Finally, consider the algebras $(E_8, \frac{15u+1}{29}, 1)$. The arguments for determining the Calabi-Yau dimension by [8, thm. 6.1] are very similar to the previous case of E_7 . The relevant invariants for type E_8 are: The frequency $f = \frac{15u+1}{29}$, the Coxeter number $h_{E_8} = 30$ with its variant $h_{E_8}^* = h_{E_8}/2 = 15$, and $m_{E_8} = h_{E_8} - 1 = 29$. Again, both parts of [8, thm. 6.1] can apply. In [8, thm. 6.1(1)] the

FIGURE 21. The indecomposable modules x_i in Dynkin type E

Calabi-Yau dimension d satisfies $d \equiv 1 - 15^{-1} \pmod{15u+1}$ and $0 < d \leq 15u+1$. Clearly, $d = u+1$ has these properties, and hence the Calabi-Yau dimension is $u+1$, as desired.

In [8, thm. 6.1(2)] the Calabi-Yau dimension d has the form $d = 2r+1$ where $r \equiv -30^{-1} \pmod{15u+1}$ and $0 \leq r < 15u+1$. As before, when [8, thm. 6.1(2)] applies then $2 \nmid f = \frac{15u+1}{29}$ which implies that u is even. Setting $r = \frac{u}{2}$ we get that $30r \equiv 15u \equiv -1 \pmod{15u+1}$, i.e. the Calabi-Yau dimension of the stable category in this case is also $2r+1 = u+1$, as desired.

u-cluster tilting object. To find a u -cluster tilting object X , as in type D , let X be the direct sum of the indecomposable projective modules and the modules $x_1, \dots, x_6, x_7, x_8$ whose positions in the stable AR quiver of the selfinjective algebra are given by Figure 21, with the convention that the summands x_7 and x_8 only occur in types E_7 and E_8 as relevant.

As all algebras in this theorem are standard, their stable module categories are equivalent to the mesh categories of their AR quivers. In particular, whether for two objects v, w we have $\underline{\text{Hom}}(v, w) \neq 0$ is completely determined by the mesh relations.

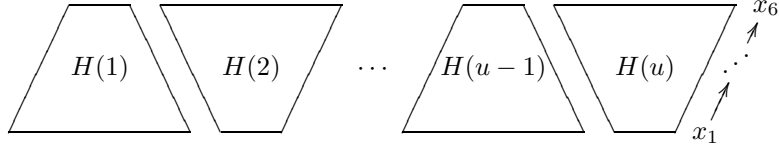
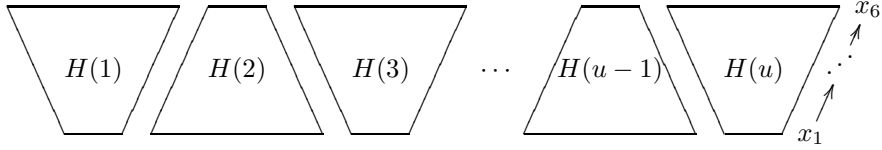
For type E and our special choice of object X , we get the following description of the indecomposable objects t such that $\underline{\text{Hom}}(X, t) \neq 0$ directly from the mesh relations (we leave the details of the straightforward, though tedious, verification of these facts to the reader).

In type E_6 , the objects t are precisely the ones lying in a trapezium with X as left side and $\tau \Sigma X$ as right side; i.e. a trapezium with X as left side, with top side containing 4 vertices and bottom side containing 8 vertices (recall from Section 2 that Σ is acting by shifting 6 units to the right and reflecting in the central line).

In types E_7 and E_8 the situation is different. The indecomposable objects t such that $\underline{\text{Hom}}(X, t) \neq 0$ are precisely the ones lying in a parallelogram with X as left side, and top and bottom sides containing 9 (for E_7) and 15 (for E_8) vertices, respectively.

Now we are in a position to show that our chosen object X is indeed a u -cluster tilting object. We need to describe the objects t with $\underline{\text{Hom}}(X, \Sigma^i t) \neq 0$ for some $i \in \{1, \dots, u\}$. For this purpose, let us consider the regions $H(j)$ of the stable AR quiver corresponding to indecomposable objects t for which

$$\underline{\text{Hom}}(X, \Sigma^{(u+1)-j} t) \neq 0 \quad (3)$$

FIGURE 22. The sets $H(i)$ in Dynkin type E_6 for u evenFIGURE 23. The sets $H(i)$ in Dynkin type E_6 for u odd

where j ranges through $\{1, \dots, u\}$.

For type E_6 we suppose that $u \equiv -2 \pmod{11}$. We have to distinguish the cases where u is even and odd, respectively. According to the above description, the regions $H(j)$ look as follows.

If u is even, then they tile a parallelogram with left side $\Sigma^{-u}T$ and right side τT as in Figure 22. In particular, the top and bottom sides of this parallelogram contain $\frac{u}{2} \cdot 12 = 6u$ vertices. But by [4, cor. 1.7] (cf. also the remarks at the beginning of this section), the stable category $\mathbf{stab}(E_6, \frac{6u+1}{11}, 1)$ has precisely $66 \cdot \frac{6u+1}{11} = 6(6u+1)$ indecomposable objects, i.e. the stable AR quiver (of tree class E_6) is identified after $6u+1$ steps. Hence it follows that

$$\underline{\mathbf{Hom}}(X, \Sigma^i t) = 0 \text{ for all } i = 1, \dots, u \text{ if and only if } t \in \text{add } X.$$

A very similar argument shows that also

$$\underline{\mathbf{Hom}}(t, \Sigma^i X) = 0 \text{ for all } i = 1, \dots, u \text{ if and only if } t \in \text{add } X.$$

Thus we have shown that our chosen object X is indeed a u -cluster tilting object in the stable module category $\mathbf{stab}(E_6, \frac{6u+1}{11}, 1)$.

If u is odd, then the regions $H(j)$ for j ranging through $\{1, \dots, u\}$ tile a trapezium with left side $\Sigma^{-u}X$ and right side τX as in Figure 23. In particular, the top side of this trapezium contains $\frac{u-1}{2} \cdot 12 + 8 = 6u + 2$ vertices, and the bottom side contains $6u - 2$ vertices. In total, this trapezium then contains $36u$ vertices (e.g. note that each of the u smaller trapeziums with top and bottom sides of length 4 and 8 contains 36 vertices). But by [4, cor. 1.7] (cf. also the remarks at the beginning of this section), the stable category $\mathbf{stab}(E_6, \frac{6u+1}{11}, 2)$ has precisely $6(6u+1) = 36u + 6$ indecomposable objects. Thus, the above trapezium fills precisely the region between the parts which become identified in the stable AR quiver. Now we can argue as above to deduce that X is indeed a u -cluster tilting object in $\mathbf{stab}(E_6, \frac{6u+1}{11}, 2)$.

For types E_7 and E_8 we suppose that $u \equiv -2 \pmod{17}$ and $u \equiv -2 \pmod{29}$, respectively. Similarly to the above considerations in type E_6 when u is even, the regions $H(j)$ of indecomposable objects X satisfying equation (3) tile a parallelogram with top and bottom rows containing $9u$ (for E_7) and $15u$ (for E_8) vertices. A sketch would resemble Figure 19. On the other hand, again by

[4, cor. 1.7], the number of indecomposable objects for the stable categories $\text{stab}(E_7, \frac{9u+1}{17}, 1)$ and $\text{stab}(E_8, \frac{15u+1}{29}, 1)$ occurring in the theorem are $119 \cdot \frac{9u+1}{17} = 7(9u+1)$ and $232 \cdot \frac{15u+1}{29} = 8(15u+1)$, respectively. Hence, the AR quivers of these stable categories are identified after $9u+1$ (for E_7) and $15u+1$ units (for E_8). From the sizes of the parallelograms given above it then follows (just as in the previous cases) that X is a u -cluster tilting object of the relevant stable categories in types E_7 and E_8 .

The desired fact that the stable endomorphism algebra $\underline{\text{End}}(X)$ is kE_n is proved verbatim as in type D above.

Vanishing of negative self-extensions. We must show $\underline{\text{Hom}}(X, \Sigma^{-i}X) = 0$ for $i = 1, \dots, u-1$. We have described above the regions in the stable AR quiver where the modules t are located for which $\underline{\text{Hom}}(X, t) \neq 0$; let us again denote them by $H(X)$. These regions $H(X)$ are certain trapeziums (for E_6) or parallelograms (for E_7 and E_8).

For type E_6 we get a situation for which a sketch would resemble the one above. The situation for E_7 and E_8 is completely analogous, but using parallelograms instead of trapeziums.

As in type D , the only way we could fail to get $\underline{\text{Hom}}(X, \Sigma^{-i}X) = 0$ would be if we took i so large that $\Sigma^i X$ made it all the way around the stable AR quiver and reached the forbidden region from the right.

However, this does not happen: In fact, left of $\Sigma^{-(u-1)}X$ we have the parallelogram (resp. trapezium) $\Sigma^{-u}H(X)$ before objects get identified in the stable AR quiver. Hence $\underline{\text{Hom}}(X, \Sigma^{-i}X) = 0$ for $i = 1, \dots, u-1$ as desired. Note that it is crucial that the maximum value for i here is $u-1$; of course, we have that $\underline{\text{Hom}}(X, \Sigma^{-u}X) \neq 0$. \square

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